Perturbative formulation for nonlinear chromaticity of circular accelerators

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The nonlinear chromaticity plays an important part in the dynamics of a particle at far off-momentum in a strong focusing circular accelerator. We derived the exact perturbative formula of the nonlinear dispersion function of a ring accelerator and gave explicit expressions of higher-order terms. Using the perturbative formula for the dispersion function, we derived the higher-order expressions for the nonlinear chromaticity up to the third order. We numerically estimated the nonlinear chromaticity of the SPring-8 storage ring, and it showed fairly good agreement with the measurement.

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I. INTRODUCTION

A high-energy storage ring dedicated to a high-brilliance light source can be characterized by low emittance. The ring optics needs a strong focusing force to achieve this low emittance. Hence, to correct the large chromaticity of the storage ring, one must install strong sextupole magnets, which inevitably enhance the nonlinearity of the optics. However, the beam lifetime is another important figure of merit of a high-brilliance light source. A larger momentum acceptance is then necessary to achieve a longer beam lifetime. To enlarge the momentum acceptance, one has to know the dynamics of a particle with large momentum deviation, where the nonlinearity of the optics is very strong. We should know the nonlinear behavior of the chromaticity because the chromaticity plays an important role in determining the momentum acceptance.

To the best of our knowledge [1–8], no exact formula of the higher-order terms of the nonlinear chromaticity is available. This is partly because we did not have a precise formula to calculate the higher-order dispersion function, which was recently derived in our previous paper [9]. Although the recipe to compute the higher-order dispersion function recurrently is given in Ref. [1], an explicit formula higher than the second order cannot be found. We derived the complete perturbative formula for the higher-order dispersion function to the fourth order and determined its validity by comparing the numerical estimation with the measurement at the SPring-8 storage ring [9]. Furthermore, in some of the aforementioned research [1–3], the higher-order modulations of the optics functions due to the energy deviation were not correctly taken into account. Thus, the higher-order terms were calculated inconsistently. The purpose of this paper is to derive a precise perturbative formula for the higher-order terms of nonlinear chromaticity. We establish the higher-order formula by reexamining the transfer matrix formulation for a particle dynamics with off-energy.

In Sec. II, after a brief explanation for the formulation of particle dynamics in a ring accelerator, we derive the perturbative formula of the nonlinear chromaticity. In Sec. III, we numerically estimate the nonlinear chromaticity of the SPring-8 storage ring and compare it with an experimental measurement.

II. FORMULATION OF NORLINEAR CHROMATICITY

A. Preliminary

In this paper, we assume the following properties for the composing magnets.

(1) There is no vertical curvature.
(2) There is no skew magnetic element.
(3) All magnets have separate functions.
(4) All magnets are approximated to have a hard edge.

The Hamiltonian H, which describes the motion of a particle in such a ring, is given by [9,10]

$$H(x,p_x,y,p_y,s) = -[1+K_x(s)x]\sqrt{(1+\delta)^2-p_x^2-p_y^2} + \frac{1}{2}[1+K_x(s)x]^2 + \sum_{n=0}\frac{g_n(s)}{(n+2)!}\sum_{m=0}^{[n/2]+1}(-)^m \times\binom{n+2}{m}x^{n+2-2m}y^{2m}, \quad (1)$$

where δ is the fractional deviation of the momentum,

$$\delta = \frac{p-p_0}{p_0}, \quad (2)$$

with the nominal momentum p_0 and s is the path length along the reference orbit. In addition, K_x is the horizontal curvature and g_n's are the strengths of multipole magnets, respectively. Furthermore, we used standard mathematical notations, such as the Gauss symbol $[\cdot]$ and the binomial coefficient $\binom{\cdot}{\cdot}$. Note that the momenta $p_{x,y}$ are normalized by the nominal momentum p_0.

The presence of the linear term with respect to x in Eq. (1) implies that with nonzero δ, the trivial solution $x\equiv 0$ can never satisfy the equation of horizontal motion derived from the Hamiltonian. A dispersion function is introduced to ex-

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tract the closed orbit for an off-momentum particle. The full order nonlinear solutions of the off-momentum trajectory $x_\epsilon(s)$ and the conjugate momentum $p_\epsilon(s)$ satisfy

$$x'_\epsilon = (1 + K_x x_\epsilon)\frac{p_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}}, \tag{3}$$

$$p'_\epsilon = K_x[\sqrt{(1+\delta)^2 - p_\epsilon^2} - 1] - K_x^2 x_\epsilon - \sum_{n=0} \frac{g_n}{(n+1)!} x_\epsilon^{n+1}. \tag{4}$$

These can be solved by the perturbative expansion

$$x_\epsilon(s) = \delta\eta(s) = \delta\sum_{n=0} \delta^n \eta_n(s), \tag{5}$$

$$p_\epsilon(s) = \delta\zeta(s) = \delta\sum_{n=0} \delta^n \zeta_n(s). \tag{6}$$

The explicit expressions of the higher-order terms of the dispersion function up to the fourth order are given in Ref. [9].

In the following, we consider the linear motion of the betatron oscillation around the off-momentum trajectory $[x_\epsilon(s), p_\epsilon(s)]$. To shift the origin of phase space to (x_ϵ, p_ϵ), we perform a canonical transformation from (x, p_x) to (x_β, p_{x_β}), which is generated by

$$F_2(x, p_{x_\beta}, s) = [x - x_\epsilon(s)][p_{x_\beta} + p_\epsilon(s)]. \tag{7}$$

The generating function $F_2(x, p_{x_\beta}, s)$ yields the transformation equations

$$x = x_\beta + x_\epsilon(s), \tag{8}$$

$$p_x = p_{x_\beta} + p_\epsilon(s), \tag{9}$$

and

$$H_\beta = H + \frac{\partial F_2}{\partial s}, \tag{10}$$

where the identity transformations for y and p_y have been suppressed. Then, up to the second order on x_β, y, p_{x_β}, and p_y, the Hamiltonian H_β is given by

$$\begin{aligned} H_\beta(x_\beta, p_{x_\beta}, y, p_y, s) = {} & \frac{1 + K_x(s)x_\epsilon(s)}{2\sqrt{(1+\delta)^2 - p_\epsilon^2(s)}}\left[\frac{(1+\delta)^2}{(1+\delta)^2 - p_\epsilon^2(s)} p_{x_\beta}^2 \right. \\ & \left. + p_y^2\right] + \frac{K_x(s)p_\epsilon(s)}{\sqrt{(1+\delta)^2 - p_\epsilon^2(s)}} x_\beta p_{x_\beta} \\ & + \frac{1}{2}K_x^2(s)x_\beta^2 + \frac{1}{2}\left(\sum_{n=0}\frac{g_n(s)}{n!}x_\epsilon^n(s)\right) \\ & \times (x_\beta^2 - y^2). \end{aligned} \tag{11}$$

The equations of motion obtained from the above Hamiltonian are

$$\frac{d}{ds}\begin{pmatrix} x_\beta \\ p_{x_\beta} \end{pmatrix} = \mathbf{T}_x(s)\begin{pmatrix} x_\beta \\ p_{x_\beta} \end{pmatrix}, \tag{12}$$

$$\frac{d}{ds}\begin{pmatrix} y \\ p_y \end{pmatrix} = \mathbf{T}_y(s)\begin{pmatrix} y \\ p_y \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} \mathbf{T}_x(s) &= \begin{pmatrix} \dfrac{K_x p_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} & \dfrac{(1+\delta)^2(1 + K_x x_\epsilon)}{[(1+\delta)^2 - p_\epsilon^2]^{3/2}} \\ -K_x^2 - \sum_{n=0}\dfrac{g_n}{n!}x_\epsilon^n & -\dfrac{K_x p_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} \end{pmatrix} \\ &\equiv \begin{pmatrix} A_x(s) & B_x(s) \\ -C_x(s) & -A_x(s) \end{pmatrix}, \end{aligned} \tag{14}$$

$$\mathbf{T}_y(s) = \begin{pmatrix} 0 & \dfrac{1 + K_x x_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} \\ \sum_{n=0}\dfrac{g_n}{n!}x_\epsilon^n & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & B_y(s) \\ -C_y(s) & 0 \end{pmatrix}. \tag{15}$$

After performing the transformation

$$\begin{pmatrix} z \\ p_z \end{pmatrix} = \mathbf{U}_z(s)\begin{pmatrix} \bar{z} \\ p_{\bar{z}} \end{pmatrix}, \tag{16}$$

$$\mathbf{U}_z(s) = \begin{pmatrix} B_z^{1/2} & 0 \\ B_z^{-1/2}\left(\frac{1}{2}B_z^{-1}B'_z - A_z\right) & B_z^{-1/2} \end{pmatrix}, \tag{17}$$

for $z = x, y$, respectively, we have "Hill's equation"

$$\frac{d}{ds}\begin{pmatrix} \bar{z} \\ p_{\bar{z}} \end{pmatrix} = \mathbf{T}_{\bar{z}}(s)\begin{pmatrix} \bar{z} \\ p_{\bar{z}} \end{pmatrix} \tag{18}$$

with

$$\mathbf{T}_{\bar{z}}(s) = \begin{pmatrix} 0 & 1 \\ -G_z(s) & 0 \end{pmatrix}, \tag{19}$$

$$G_z(s) = B_z C_z - A'_z + \tfrac{1}{2}(\ln B_z)'' - \left[A_z - \tfrac{1}{2}(\ln B_z)'\right]^2. \tag{20}$$

According to the standard manner of Hill's equation, we can construct the transfer matrix $\mathbf{M}_{\bar{z}}(s_1|s_0)$,

$$\left.\begin{pmatrix} \bar{z} \\ p_{\bar{z}} \end{pmatrix}\right|_{s_1} = \mathbf{M}_{\bar{z}}(s_1|s_0)\begin{pmatrix} \bar{z} \\ p_{\bar{z}} \end{pmatrix}_{s_0}, \tag{21}$$

which can be represented by the Twiss parameters as [11]

$$\mathbf{M}_{\bar{z}}(s_1|s_0) = \begin{pmatrix} \sqrt{\frac{\beta_{\bar{z}}(s_1)}{\beta_{\bar{z}}(s_0)}}\left[\cos \psi_{\bar{z}}\Big|_{s_0}^{s_1} + \alpha_{\bar{z}}(s_0)\sin \psi_{\bar{z}}\Big|_{s_0}^{s_1}\right] & \sqrt{\beta_{\bar{z}}(s_1)\beta_{\bar{z}}(s_0)}\sin \psi_{\bar{z}}\Big|_{s_0}^{s_1} \\ -\frac{\alpha_{\bar{z}}(s_1)-\alpha_{\bar{z}}(s_0)}{\sqrt{\beta_{\bar{z}}(s_1)\beta_{\bar{z}}(s_0)}}\cos \psi_{\bar{z}}\Big|_{s_0}^{s_1} - \frac{1+\alpha_{\bar{z}}(s_1)\alpha_{\bar{z}}(s_0)}{\sqrt{\beta_{\bar{z}}(s_1)\beta_{\bar{z}}(s_0)}}\sin \psi_{\bar{z}}\Big|_{s_0}^{s_1} & \sqrt{\frac{\beta_{\bar{z}}(s_0)}{\beta_{\bar{z}}(s_1)}}\left[\cos \psi_{\bar{z}}\Big|_{s_0}^{s_1} - \alpha_{\bar{z}}(s_1)\sin \psi_{\bar{z}}\Big|_{s_0}^{s_1}\right] \end{pmatrix} \tag{22}$$

with the betatron phase

$$\psi_{\bar{z}}\Big|_{s_0}^{s_1} = \int_{s_0}^{s_1} \frac{ds}{\beta_{\bar{z}}(s)}. \tag{23}$$

Then, the one turn transfer matrix at s_0 is given by

$$\mathbf{M}_{\bar{z}}(s_0+L|s_0) = \begin{pmatrix} \cos \mu_{\bar{z}} + \alpha_{\bar{z}}(s_0)\sin \mu_{\bar{z}} & \beta_{\bar{z}}(s_0)\sin \mu_{\bar{z}} \\ -\gamma_{\bar{z}}(s_0)\sin \mu_{\bar{z}} & \cos \mu_{\bar{z}} - \alpha_{\bar{z}}(s_0)\sin \mu_{\bar{z}} \end{pmatrix}, \tag{24}$$

where L is the circumference of the ring. Here, we have used the periodicity of the Twiss parameters and defined the phase advance over the circumference

$$\mu_{\bar{z}} = \int_{s_0}^{s_0+L} \frac{ds}{\beta_{\bar{z}}(s)}. \tag{25}$$

From Eq. (24), one can relate the tune $\nu_{\bar{z}} \equiv \mu_{\bar{z}}/(2\pi)$ to the transfer matrix as

$$\cos \mu_{\bar{z}} = \cos 2\pi\nu_{\bar{z}} = \frac{1}{2}\mathrm{Tr}\ \mathbf{M}_{\bar{z}}(s_0+L|s_0). \tag{26}$$

Now, we perturbatively calculate the nonlinear chromaticity based on the defining equation of the tune (26) [7]. Hereafter, for simplicity, we omit the suffix denoting the coordinates such as x, y if they are not necessary. The expansion of the phase advance μ with respect to the momentum deviation δ,

$$\mu = \sum_{n=0} \delta^n \mu_n, \tag{27}$$

gives

$$\cos \mu = \sum_{n=0} \delta^n \chi_n, \tag{28}$$

where

$$\chi_0 = \cos \mu_0, \tag{29}$$

$$\chi_1 = -\mu_1 \sin \mu_0, \tag{30}$$

$$\chi_2 = -\mu_2 \sin \mu_0 - \frac{1}{2}\mu_1^2 \cos \mu_0, \tag{31}$$

$$\chi_3 = -\left(\mu_3 - \frac{1}{6}\mu_1^3\right)\sin \mu_0 - \mu_2\mu_1 \cos \mu_0, \tag{32}$$

and so on. The transfer matrix $\mathbf{M}$ is also expanded with respect to δ as

$$\mathbf{M}(s_0+L|s_0) = \sum_{n=0} \delta^n \mathbf{M}_n(s_0+L|s_0). \tag{33}$$

Comparing Eqs. (28) and (23), we can obtain the representation of χ_n in terms of the transfer matrices,

$$\chi_n = \frac{1}{2}\mathrm{Tr}\ \mathbf{M}_n(s_0+L|s_0). \tag{34}$$

Now, we derive the explicit representations of the higher-order transfer matrices $\mathbf{M}_n(s_0+L|s_0)$ in terms of the perturbative expansion of the impact matrix $\mathbf{T}(s)$. Dividing the distance between s_1 and s_0 ($s_1 > s_0$) into an infinite number of infinitesimal ones, we can write the transfer matrix as the infinite product of the infinitesimal ones,

$$\mathbf{M}(s_1|s_0) = \lim_{m\to\infty} \prod_{i=0}^{m-1} \mathbf{M}(t_{i+1}|t_i) \tag{35}$$

with $t_i = s_0 + [(s_1 - s_0)/m]i$. From Eq. (18), the transfer matrix corresponding to the infinitesimal interval is given by

$$\mathbf{M}(t_{i+1}|t_i) = \mathbf{I} + (t_{i+1} - t_i)\mathbf{T}(t_i), \tag{36}$$

where $\mathbf{I}$ is the identity matrix,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

Expanding $\mathbf{T}$ in terms of δ,

$$\mathbf{T}(s) = \sum_{n=0} \delta^n \mathbf{T}_n(s), \tag{38}$$

and gathering the terms order by order in the matrix product (35), we can derive the perturbative expansion of the transfer matrix $\mathbf{M}(s_1|s_0)$. At the zeroth order, we have the unperturbed transfer matrix,

$$\mathbf{M}_0(s_1|s_0) = \lim_{m\to\infty}\prod_{i=0}^{m-1}[\mathbf{I} + (t_{i+1} - t_i)\mathbf{T}_0(t_i)]. \quad (39)$$

As an example of a drift space we know,

$$\mathbf{T}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (40)$$

Thus, due to the nilpotency of $\mathbf{T}_0$,

$$\mathbf{M}_0(s_1|s_0) = \lim_{m\to\infty}\left[\mathbf{I} + \sum_{i=0}^{m-1}(t_{i+1} - t_i)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & s_1 - s_0 \\ 0 & 1 \end{pmatrix}, \quad (41)$$

which is the transfer matrix of the drift space of the interval $s_1 - s_0$. For another fundamental element, a quadrupole magnet, we can show the equivalence of the product representation (39) with the usual matrix as well. The higher-order transfer matrices in the momentum deviation δ are calculated in the following subsections.

B. Representation of linear chromaticity

Picking up the linear terms $\mathbf{T}_1(s_1)$ in the infinite product form of the transfer matrix, Eq. (35) with Eq. (36), we obtain the first-order one-turn transfer matrix $\mathbf{M}_1(s_0+L|s_0)$,

$$\mathbf{M}_1(s_0 + L|s_0) = \lim_{m\to\infty}\sum_{k=0}^{m-1}\prod_{i=k+1}^{m-1}[\mathbf{I} + (t_{i+1} - t_i)\mathbf{T}_0(t_i)](t_{k+1} - t_k) \times\mathbf{T}_1(t_k)\prod_{j=0}^{k-1}[\mathbf{I} + (t_{j+1} - t_j)\mathbf{T}_0(t_j)], \quad (42)$$

which, with the help of Eq. (39), can be casted into the integral form

$$\mathbf{M}_1(s_0 + L|s_0) = \int_{s_0}^{s_0+L} ds_1 \mathbf{M}_0(s_0 + L|s_1)\mathbf{T}_1(s_1)\mathbf{M}_0(s_1|s_0). \quad (43)$$

Because $\mathbf{M}_0$ has the periodicity of a circumference L,

$$\mathbf{M}_0(s_1|s_0) = \mathbf{M}_0(s_1 + L|s_0 + L). \quad (44)$$

The trace of $\mathbf{M}_1$ is written by that of the product of the zeroth-order one turn matrix $\mathbf{M}_0(s_1+L|s_1)$ with the first-order impact matrix $\mathbf{T}_1(s_1)$,

$$\mathrm{Tr}\mathbf{M}_1(s_0 + L|s_0) = \int_{s_0}^{s_0+L} ds_1 \mathrm{Tr}[\mathbf{M}_0(s_1 + L|s_1)\mathbf{T}_1(s_1)]. \quad (45)$$

Note that the momentum modulation of the phase advance is represented by the product of the unperturbed transfer matrix and the first-order impact matrix. As seen in the following, the higher-order representations of the nonlinear chromaticities are also given by products of the zeroth-order transfer matrices and the higher-order impact ones. In other words, in our formulation of the nonlinear chromaticity, higher-order aberrations of the lattice functions, especially the betatron function, do not appear. Aberrations of beta function are omitted in papers [1–3].

Using the first-order representation of the instant transfer matrix T with respect to the momentum deviation [see Eq. (19)]

$$\mathbf{T}_1(s_1) = \begin{pmatrix} 0 & 0 \\ -G_1(s_1) & 0 \end{pmatrix}, \quad (46)$$

where G_1 is the first-order coefficient of G (20), and

$$\mathbf{M}_0(s_1 + L|s_1) = \begin{pmatrix} \cos\mu_0 + \alpha(s_1)\sin\mu_0 & \beta(s_1)\sin\mu_0 \\ -\gamma(s_1)\sin\mu_0 & \cos\mu_0 - \alpha(s_1)\sin\mu_0 \end{pmatrix},$$

we obtain

$$\mathrm{Tr}\mathbf{M}_1(s_0 + L|s_0) = -\sin\mu_0\int_{s_0}^{s_0+L} ds_1\beta(s_1)G_1(s_1). \quad (47)$$

Here, we suppress the suffix 0 of the unperturbed lattice functions. Then, taking Eqs. (26), (28), and (30) into account, the first-order phase variation due to the momentum deviation is given by

$$\mu_1 = \frac{1}{2}\int_{s_0}^{s_0+L} ds_1\beta(s_1)G_1(s_1). \quad (48)$$

For example, by inserting the explicit expression of G_1 for horizontal motion,

$$G_1 = (K_x\eta_0 - 1)(K_x^2 + g_0) + g_1\eta_0 - (K_x\eta_0')' + \tfrac{1}{2}(K_x\eta_0)'', \quad (49)$$

into Eq. (48) and by partially integrating the resultant representation, we find that the linear chromaticity $\xi_1 \equiv \mu_1/(2\pi)$ is given by

$$\xi_1 = \frac{1}{4\pi}\int_{s_0}^{s_0+L} ds_1[-\beta_x(K_x^2 + g_0 - g_1\eta_0) - 2\alpha_x K_x\eta_0' + \gamma_x K_x\eta_0], \quad (50)$$

which is the very formula for the linear chromaticity [5,7,12]. Here, we have used the well-known defining identities

$$\alpha_x = -\tfrac{1}{2}\beta_x' \quad (51)$$

$$\gamma_x = (K_x^2 + g_0)\beta_x + \tfrac{1}{2}\beta_x''. \quad (52)$$

The expression for the vertical case can be found in Appendix B.

C. Representation of quadratic chromaticity

Next, we calculate the second-order variation of the phase advance μ_2 with respect to the momentum deviation δ. The second-order transfer matrix consists of two terms; one is the second-order impact $\mathbf{T}_2$ of the equation of motion, and the other is the quadratic product of the first-order impact $\mathbf{T}_1$, i.e.,

$$\mathbf{M}_2(s_0+L|s_0) = \int_{s_0}^{s_0+L} ds_1 \mathbf{M}_0(s_0+L|s_1)\mathbf{T}_2(s_1)\mathbf{M}_0(s_1|s_0) + \int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \mathbf{M}_0(s_0+L|s_2)\mathbf{T}_1(s_2) \times \mathbf{M}_0(s_2|s_1)\mathbf{T}_1(s_1)\mathbf{M}_0(s_1|s_0).$$

The trace of the former term can be calculated in a similar way to the first-order case, so that we have

$$\int_{s_0}^{s_0+L} ds_1 \mathrm{Tr}[\mathbf{M}_0(s_1+L|s_1)\mathbf{T}_2(s_1)] = -\sin\,\mu_0 \int_{s_0}^{s_0+L} ds_1 \beta(s_1) G_2(s_1), \tag{53}$$

where G_2 is the second-order coefficient of G (20), whose explicit form after partial integration is given in Appendix B. The slightly complex calculation brings the trace of the quadratic product matrix of the first-order impact into the following form:

$$\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \mathrm{Tr}[\mathbf{M}_0(s_1+L|s_2)\mathbf{T}_1(s_2)\mathbf{M}_0(s_2|s_1)\mathbf{T}_1(s_1)] = \frac{1}{2}\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \Big[\cos\Big(\mu_0 - 2\psi\Big|_{s_1}^{s_2}\Big) - \cos\,\mu_0\Big] \times \beta(s_2)G_1(s_2)\beta(s_1)G_1(s_1).$$

The term proportional to $\cos\,\mu_0$ in the integrand of the above equation can be easily rewritten as the square of the single integral,

$$\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \beta(s_2)G_1(s_2)\beta(s_1)G_1(s_1) = \frac{1}{2}\left[\int_{s_0}^{s_0+L} ds_1 \beta(s_1)G_1(s_1)\right]^2,$$

which, taking Eq. (48) into account, is completely canceled out by the product of the first-order phase variation μ_1 in Eq. (31). We perform the Fourier transformation of G_1 to integrate the term proportional to $\cos(\mu_0 - 2\psi|_{s_1}^{s_2})$. Because

$$ds_1 = \beta(s_1)d\varphi(s_1) \tag{54}$$

with

$$\varphi(s_1) \equiv \int_{s_0}^{s_1} \frac{ds}{\beta(s)}, \tag{55}$$

we find

$$\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \cos\left(\mu_0 - 2\psi\Big|_{s_1}^{s_2}\right)\beta(s_2)G_1(s_2)\beta(s_1)G_1(s_1) = \int_0^{\mu_0} d\varphi_2 \int_0^{\varphi_2} d\varphi_1 \cos(\mu_0 - 2\varphi_2 + 2\varphi_1) \times \beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1).$$

Using the fact that $\beta^2(\varphi)G_1(\varphi)$ has the period μ_0, we expand it as

$$\beta^2(\varphi)G_1(\varphi) = \frac{1}{2}a_1(0) + \sum_{n=1}^{\infty}\left[a_1(n)\cos\left(\frac{2\pi n}{\mu_0}\varphi\right) + b_1(n)\sin\left(\frac{2\pi n}{\mu_0}\varphi\right)\right], \tag{56}$$

where

$$a_1(n) = \frac{2}{\mu_0}\int_0^{\mu_0} d\varphi\, \cos\left(\frac{2\pi n}{\mu_0}\varphi\right)\beta^2(\varphi)G_1(\varphi) \quad (n=0,1,2,\ldots), \tag{57}$$

$$b_1(n) = \frac{2}{\mu_0}\int_0^{\mu_0} d\varphi\, \sin\left(\frac{2\pi n}{\mu_0}\varphi\right)\beta^2(\varphi)G_1(\varphi) \quad (n=1,2,\ldots). \tag{58}$$

Now, by performing the double integration of the phases as shown in Appendix D, we have

$$\int_0^{\mu_0} d\varphi_2 \int_0^{\varphi_2} d\varphi_1 \cos(\mu_0 - 2\varphi_2 + 2\varphi_1) \times \beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1) = \sin\,\mu_0\left[\frac{1}{8}\mu_0 a_1^2(0) + \sum_{n=1}^{\infty}\frac{\mu_0^3}{4(\mu_0^2 - \pi^2 n^2)} \times \{a_1^2(n) + b_1^2(n)\}\right].$$

After all of these calculations are made, the second-order chromaticity $\xi_2 \equiv \mu_2/(2\pi)$ is given by

$$\xi_2 = \frac{1}{4\pi}\left[\int_{s_0}^{s_0+L} ds_1 \beta(s_1)G_2(s_1) - \frac{1}{16}\mu_0 a_1^2(0) - \sum_{n=1}^{\infty}\frac{\mu_0^3}{8(\mu_0^2 - \pi^2 n^2)}\{a_1^2(n) + b_1^2(n)\}\right]. \tag{59}$$

Changing the variables from φ to s again, we have the integral form of the Fourier components $a_1(n)$ and $b_1(n)$ as

$$a_1(n) = \frac{2}{\mu_0}\int_{s_0}^{s_0+L} ds_1 \cos\left[\frac{2\pi n}{\mu_0}\varphi(s_1)\right]\beta(s_1)G_1(s_1) \times (n=0,1,2,\ldots), \tag{60}$$

$$b_1(n)=\frac{2}{\mu_0}\int_{s_0}^{s_0+L} ds_1 \sin\left[\frac{2\pi n}{\mu_0}\varphi(s_1)\right]\beta(s_1)G_1(s_1)$$
$$\times(n=1,2,\ldots). \qquad (61)$$

The explicit forms of G_2, $a_1(n)$, and $b_1(n)$ are given in Appendixes B and C, respectively.

D. Representation of cubic chromaticity

The one-turn transfer matrix $\mathbf{M}_3(s_0+L|s_0)$ of the third order in δ consists of three integrals,

$$\mathbf{M}_3(s_0+L|s_0)=\int_{s_0}^{s_0+L} ds_1 \mathbf{M}_0(s_0+L|s_1)\mathbf{T}_3(s_1)\mathbf{M}_0(s_1|s_0)$$
$$+\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1[\mathbf{M}_0(s_0+L|s_2)\mathbf{T}_2(s_2)$$
$$\times \mathbf{M}_0(s_2|s_1)\mathbf{T}_1(s_1)\mathbf{M}_0(s_1|s_0)+(\mathbf{T}_2\leftrightarrow\mathbf{T}_1)]$$
$$+\int_{s_0}^{s_0+L} ds_3 \int_{s_0}^{s_3} ds_2 \int_{s_0}^{s_2} ds_1$$
$$\times \mathbf{M}_0(s_0+L|s_3)\mathbf{T}_1(s_3)\mathbf{M}_0(s_3|s_2)\mathbf{T}_1(s_2)$$
$$\times \mathbf{M}_0(s_2|s_1)\mathbf{T}_1(s_1)\mathbf{M}_0(s_1|s_0),$$

where $(\mathbf{T}_2\leftrightarrow\mathbf{T}_1)$ denotes the term exchanging $\mathbf{T}_2$ for $\mathbf{T}_1$ in the preceding term. In a similar manner to the second-order case, we can transform the traces of the above matrices into the following expressions:

$$\int_{s_0}^{s_0+L} ds_1 \mathrm{Tr}[\mathbf{M}_0(s_1+L|s_1)\mathbf{T}_3(s_1)]$$
$$=-\sin\,\mu_0 \int_{s_0}^{s_0+L} ds_1 \beta(s_1)G_3(s_1),$$

$$\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \mathrm{Tr}\left[\mathbf{M}_0(s_0+L|s_2)\mathbf{T}_2(s_2)\mathbf{M}_0\left(s_2|s_1)\mathbf{T}_1(s_1)\mathbf{M}_0(s_1|s_0)+(\mathbf{T}_2\leftrightarrow\mathbf{T}_1)\right]\right.$$
$$=\frac{1}{2}\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1\left[\cos\,(\mu_0-2\psi|_{s_1}^{s_2})-\cos\,\mu_0\right][\beta(s_2)G_2(s_2)\beta(s_1)G_1(s_1)+(G_2\leftrightarrow G_1)],$$

$$\int_{s_0}^{s_0+L} ds_3 \int_{s_0}^{s_3} ds_2 \int_{s_0}^{s_2} ds_1 \mathrm{Tr}\left[\mathbf{M}_0\left(s_1+L|s_3)\mathbf{T}_1(s_3)\mathbf{M}_0(s_3|s_2)\mathbf{T}_1(s_2)\mathbf{M}_0(s_2|s_1)\mathbf{T}_1(s_1)\right]\right.$$
$$=\frac{1}{4}\int_{s_0}^{s_0+L} ds_3 \int_{s_0}^{s_3} ds_2 \int_{s_0}^{s_2} ds_1\left[\sin(\mu_0-2\psi|_{s_1}^{s_3})-\sin(\mu_0-2\psi|_{s_2}^{s_3}\right)-\sin(\mu_0-2\psi|_{s_1}^{s_2}\left.\right)+\sin\,\mu_0$$
$$\times\beta(s_3)G_1(s_3)\beta(s_2)G_1(s_2)\beta(s_1)G_1(s_1).$$

The trace of the product of $\mathbf{T}_2$ and $\mathbf{T}_1$, which are the interfering terms of the first- and second-order deviations, is converted into the Fourier series by a similar method used to obtain the second-order case,

$$\int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \mathrm{Tr}\left[\mathbf{M}_0(s_0+L|s_2)\mathbf{T}_2(s_2)\mathbf{M}_0(s_2|s_1)\mathbf{T}_1(s_1)\right.$$
$$\left.\times\mathbf{M}_0(s_1|s_0)+(\mathbf{T}_2\leftrightarrow\mathbf{T}_1)\right]$$
$$=\sin\,\mu_0\left[\frac{1}{8}\mu_0 a_2(0)a_1(0)+\sum_{n=1}^{\infty}\frac{\mu_0^3}{4(\mu_0^2-\pi^2n^2)}\{a_2(n)a_1(n)\right.$$
$$\left.+b_2(n)b_1(n)\}\right]-\frac{1}{2}\cos\,\mu_0\left[\int_{s_0}^{s_0+L} ds\beta(s)G_2(s)\right]$$
$$\times\left[\int_{s_0}^{s_0+L} ds\beta(s)G_1(s)\right]. \qquad (62)$$

Here, a_2 and b_2 are the Fourier components of $\beta(s)G_2(s)$. The term proportional to $\cos\,\mu_0$ corresponds to the part of the cross term of μ_2 and μ_1 in the third-order expression (32).

The integration of the term proportional to $\sin\,\mu_0$ in the triple product of G_1, the triple coupling between the first-order deviations, can be easily casted into the cubic product of the integral of the single integral:

$$\int_{s_0}^{s_0+L} ds_3 \int_{s_0}^{s_3} ds_2 \int_{s_0}^{s_2} ds_1 \beta(s_3)G_1(s_3)\beta(s_2)G_1(s_2)\beta(s_1)G_1(s_1)$$
$$=\frac{1}{6}\left[\int_{s_0}^{s_0+L} ds_1 \beta(s_1)G_1(s_1)\right]^3, \qquad (63)$$

which is canceled out with the cubic product of the first-order chromaticity μ_1 in Eq. (32). The other terms of the triple product of G_1 can be transformed into the Fourier series by means of the formula given in Appendix D:

$$
\begin{aligned}
&\frac{1}{4}\int_{s_0}^{s_0+L} ds_3 \int_{s_0}^{s_3} ds_2 \int_{s_0}^{s_2} ds_1 [\sin(\mu_0 - 2\psi|_{s_1}^{s_3}) - \sin(\mu_0 - 2\psi|_{s_2}^{s_3} - \sin(\mu_0 - 2\psi|_{s_1}^{s_2} \times \beta(s_3)G_1(s_3)\beta(s_2)G_1(s_2)\beta(s_1)G_1(s_1) \\
&\quad = \frac{1}{8}\mu_0^2 \cos\, \mu_0 a_1(0)\left[\frac{1}{8}a_1^2(0) + \sum_{n=1}^{\infty}\frac{\mu_0^2}{4(\mu_0^2 - \pi^2 n^2)}\{a_1^2(n) + b_1^2(n)\}\right] - \frac{1}{64}\sin\, \mu_0\left[\mu_0 a_1^3(0) + 2\sum_{n=1}^{\infty}\frac{\mu_0^3(3\mu_0^2 - \pi^2 n^2)}{(\mu_0^2 - \pi^2 n^2)^2}a_1(0)\{a_1^2(n)\right. \\
&\qquad + b_1^2(n)\} + 2\sum_{n,m=1}^{\infty}\frac{\mu_0^5\{3\mu_0^2 - \pi^2(n^2 + nm + m^2)\}}{(\mu_0^2 - \pi^2 n^2)(\mu_0^2 - \pi^2 m^2)\{\mu_0^2 - \pi^2(n+m)^2\}} \times \{a_1(n+m)a_1(n)a_1(m) - a_1(n+m)b_1(n)b_1(m) + b_1(n \\
&\qquad \left. + m)a_1(n)b_1(m) + b_1(n+m)b_1(n)a_1(m)\}\right].
\end{aligned}
$$

The terms proportional to cos μ_0 in the right-hand side of the above equation together with that in Eq. (62) result in the product of μ_2 and μ_1 in Eq. (32).

Combining the above results, we find the Fourier representation of the third-order chromaticity $\xi_3 \equiv \mu_3/(2\pi)$:

$$
\begin{aligned}
\xi_3 = \frac{1}{4\pi}&\left[\int_{s_0}^{s_0+L} ds_1 \beta(s_1)G_3(s_1) - \frac{\mu_0}{8}a_2(0)a_1(0) - \sum_{n=1}^{\infty}\frac{\mu_0^3}{4(\mu_0^2 - \pi^2 n^2)}\{a_2(n)a_1(n) + b_2(n)b_1(n)\} + \frac{\mu_0}{64}a_1^3(0)\right. \\
&+ \sum_{n=1}^{\infty}\frac{\mu_0^3(3\mu_0^2 - \pi^2 n^2)}{32(\mu_0^2 - \pi^2 n^2)^2}a_1(0)\{a_1^2(n) + b_1^2(n)\} + \sum_{n,m=1}^{\infty}\frac{\mu_0^5\{3\mu_0^2 - \pi^2(n^2 + nm + m^2)\}}{32(\mu_0^2 - \pi^2 n^2)(\mu_0^2 - \pi^2 m^2)\{\mu_0^2 - \pi^2(n+m)^2\}} \\
&\left. \times \{a_1(n+m)a_1(n)a_1(m) - a_1(n+m)b_1(n)b_1(m) + b_1(n+m)a_1(n)b_1(m) + b_1(n+m)b_1(n)a_1(m)\}\right]. \qquad (64)
\end{aligned}
$$

The explicit forms of G_3, $a_2(n)$, and $b_2(n)$ as well as $a_1(n)$ and $b_1(n)$ are given in Appendixes B and C, respectively. It should be emphasized that the terms proportional to cos μ_0 in the derivation of the cubic chromaticity, i.e., the product of the linear and the square charomaticities, are completely canceled out.

III. NUMERICAL CALCULATION

Now, we apply our formula of nonlinear chromaticity to the SPring-8 storage ring. By comparing the numerical results with the measurements, we can determine the validity of the formulation.

The energy of electrons circulating in the SPring-8 storage ring is 8 GeV, and the circumference was about 1436 m. The original lattice of the ring is composed of 48 unit cells of a double bend achromat. Four of the 48 cells lack bending magnets for the purpose of installing long straight sections of 30 m. In the early years of the operation of the storage ring, the quadrupole magnets in the straight cells were used to keep the 48-fold symmetry of the optics. The strong sextupole magnets are installed to cancel the chromatic effect and the nonlinearity of the optics is considerably strong.

Until now, the SPring-8 storage ring has mainly operated in three different optic configurations. One is the so called hybrid optics, whose horizontal beta function, at the dispersion free straight sections, takes high and low values alternately. The optics functions, i.e., the horizontal and vertical betatron function and the horizontal dispersion function, in the four normal cells are shown in Fig. 1. The solid (dashed) line indicates the horizontal (vertical) beta function and the dotted one the horizontal dispersion function. The left ordinate expresses the scale of the beta functions, and the right one corresponds to the dispersion function. The boxes at a bottom express magnets, the highest ones of which correspond to the quadrupole, the lowest to the bending, and the remainder to the sextupole magnets, respectively. The hybrid optics were used from March 1997, when the SPring-8 storage ring commissioning began, to July 1999. In September 1999 the second one, called the HHLV optics, whose horizontal and vertical beta functions take high and low values at all the dispersion-free straight sections, respectively, went

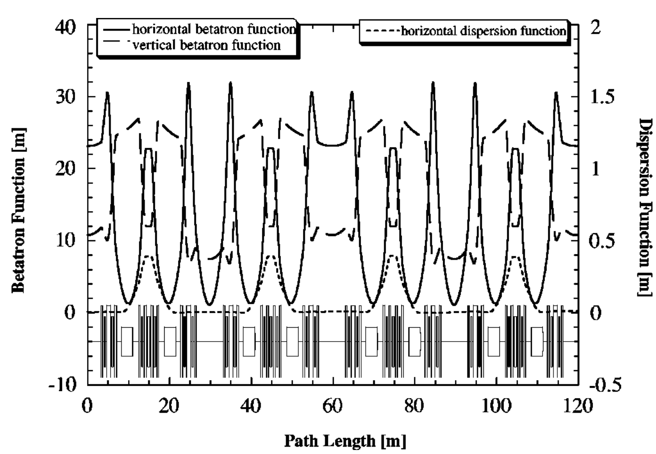

FIG. 1. Configuration of hybrid optics.

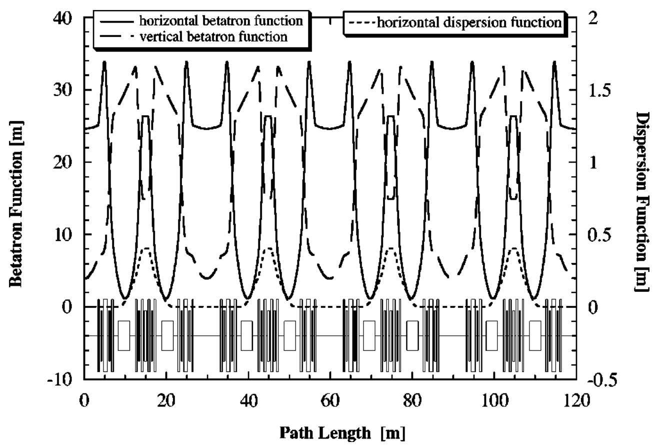

FIG. 2. Configuration of HHLV optics.

into service to use the insertion devices more effectively. Figure 2 indicates the optic functions of the HHLV. In the summer shut down of 2000, we reconstructed the storage ring, i.e., we removed the focusing magnets in the four straight cells to introduce 30-m-long magnet-free sections. We abbreviated the new optics as LSS, whose optics functions are shown in Fig. 3. Typical values of the betatron tunes and the linear chromaticities are listed in Table I.

We calculated the higher-order chromaticity of these three optics and compared the numerical results with the measurements at the SPring-8 storage ring. Figure 4 shows the measured and calculated betatron tunes of the hybrid optics as a function of the fractional momentum deviation. The circles are the measured horizontal tunes and the squares the vertical, respectively. The calculated horizontal (vertical) tunes are represented by the thick (thin) lines. In Fig. 4 the dotted lines denote the calculated tunes including only the linear chromaticities, and the dashed lines correspond to the second-order calculations. The calculated tunes including up to the third order are represented by the solid lines. The left (right) ordinate expresses the scale of the horizontal (vertical) tune.

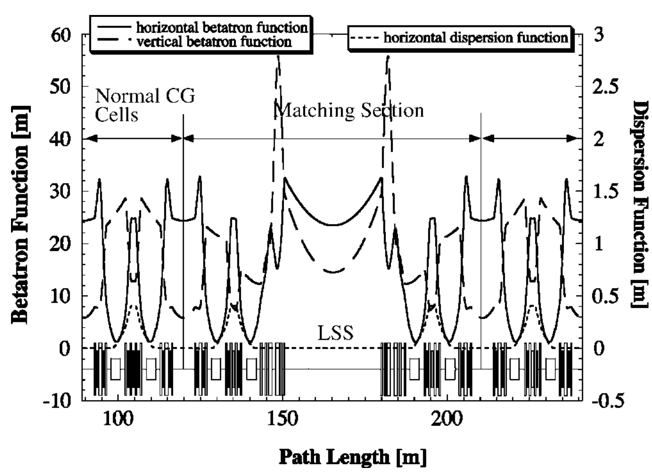

FIG. 3. Horizontal and vertical beta functions and horizontal dispersion function of LSS optics over five cells including magnet-free section.

TABLE I. List of linear optics parameters.

Optics	ν_x	ν_y	ξ_{1x}	ξ_{1y}
Hybrid	51.231	16.310	1.63	0.70
HHLV	43.159	21.358	7.06	4.09
LSS	40.199	18.350	7.18	6.37

The calculation of each element of the lattice, including the sextupole magnets, is divided into 200 pieces. We summed the harmonics of a Fourier series up to 15000, for which the series almost converged as can be seen in Fig. 5. The thick lines denote the horizontal chromaticities, and the thin ones represent those of the vertical. The solid lines indicate the second-order chromaticities, and the dotted lines correspond to the third. The left (right) ordinate expresses the scale of the second- (third-) order chromaticity. The thickness of a sextupole magnet is very important for ensuring the convergence. The higher the order of the harmonics becomes, the more times the Fourier component oscillates in magnetic elements. Hence for higher harmonics, the contributions of the pieces in an element cancel each other out so that the Fourier series converges.

The range of the momentum deviation of the hybrid optics, where we can store the beam with enough lifetime to measure the betatron tune, ranges only from −0.8% to +1.2%. One finds that even in such a narrow range of momentum deviation, the nonlinearity significantly affects on the chromaticity. But in this example with narrow momentum acceptance, one cannot find the difference between the second- and third-order perturbative approximations.

The optics change from the hybrid to the HHLV enlarged the momentum range drastically. The measured and calculated tunes of the HHLV are shown in Fig. 6. One can see in Fig. 6 that proportional to the increase in the order, the calculated tunes approach the measured values. The increase of

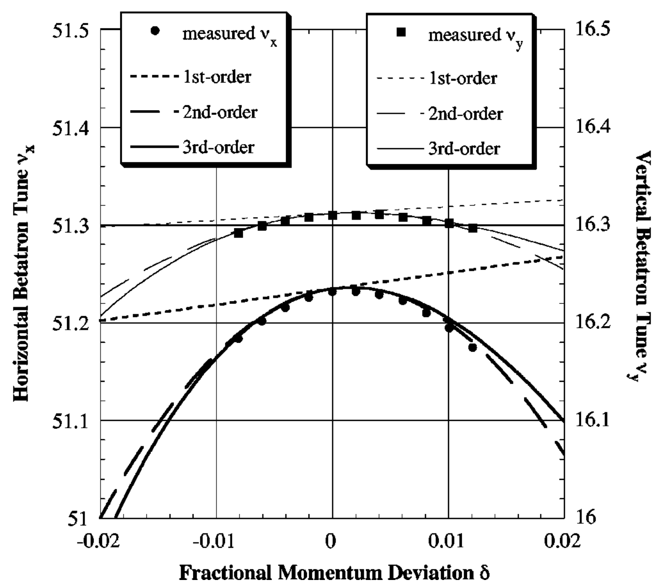

FIG. 4. Betatron tunes of hybrid optics as a function of momentum deviation.

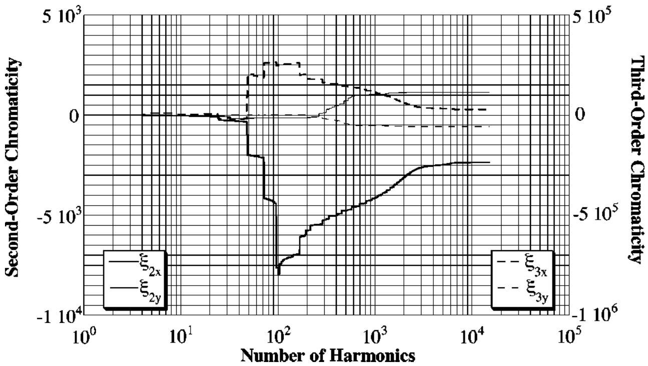

FIG. 5. Convergence of Fourier series of nonlinear chromaticity of hybrid optics on the number of the summation.

the momentum acceptance of the HHLV optics at the negative momentum deviation reaches about three times that of the hybrid. We consider the cause of the expansion of the momentum acceptance to be attributed to the sign of the second order of the horizontal chromaticity. In the HHLV optics, a particle with a large momentum deviation keeps away from an integer resonance line. However, in the hybrid, it approaches the line.

As a final illustration, Fig. 7 displays the tunes of the LSS optics. Comparing Fig. 7 with Fig. 6, we find that the nonlinearity of the LSS optics is stronger than that of the HHLV optics. Due to the stronger nonlinearity, the momentum acceptance of the LSS optics seems to become narrower than the HHLV.

IV. SUMMARY AND CONCLUSIONS

We derived the perturbative formula for the higher-order terms of nonlinear chromaticity of a circular accelerator up

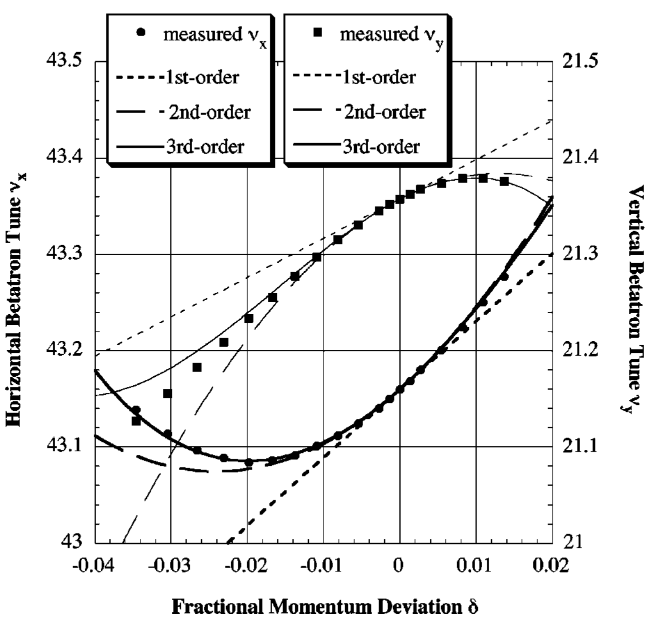

FIG. 6. Betatron tunes of HHLV optics as a function of momentum deviation.

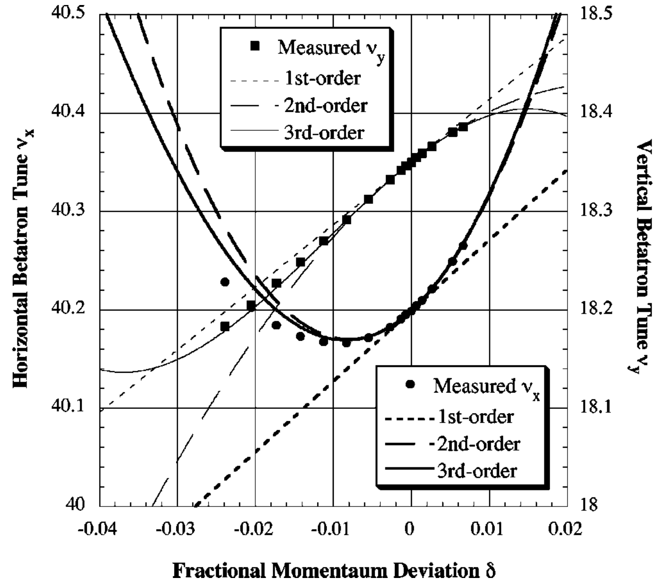

FIG. 7. Betatron tunes of LSS optics as a function of momentum deviation.

to the third order. We established a canonical equation of an off-momentum particle motion based on the full order Hamiltonian with respect to momentum deviation. Using a transfer matrix formulation derived from the equations, we gave a perturbative representation of the nonlinear chromaticity, which looks like a Feynman integral. In the formulation, we found that the Fourier expansion with respect to the lattice periodicity has great effectiveness in extracting the higher-order formula of the nonlinear chromaticity from the transfer matrices. As an example, we calculated the nonlinear chromaticities of the SPring-8 storage ring with three different optics, whose numerical results agree fairly well with the measurements.

As found in the SPring-8 storage ring, the momentum acceptance is very sensitive to the higher-order nonlinearity of the chromaticity. Higher-order terms of the nonlinear chromaticity are thus indispensable for the dynamics of a particle motion with a large momentum deviation.

ACKNOWLEDGMENT

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APPENDIX A: EXPANSION OF INFINITESIMAL TRANSFORMATION

Here, we give the explicit representation of the expansion of the contact transformation $T_z(s)(z=x,y)$ with respect to the momentum deviation δ.

(1) *Zeroth order*. For horizontal motion,

$$A_{x,0}=0,$$

$$B_{x,0}=1,$$

$$C_{x,0} = K_x^2 + g_0.$$

For vertical motion,

$$B_{y,0} = 1,$$

$$C_{y,0} = -g_0.$$

(2) *First order.*

$$A_{x,1} = K_x \eta_0',$$

$$B_{x,1} = -1 + K_x \eta_0,$$

$$C_{x,1} = g_1 \eta_0,$$

$$B_{y,1} = -1 + K_x \eta_0,$$

$$C_{y,1} = -g_1 \eta_0.$$

(3) *Second order.*

$$A_{x,2} = K_x \eta_1' - K_x^2 \eta_0 \eta_0',$$

$$B_{x,2} = 1 - K_x(\eta_0 - \eta_1) + \tfrac{3}{2}\eta_0'^2$$

$$C_{x,2} = \tfrac{1}{2} g_2 \eta_0^2 + g_1 \eta_1,$$

$$B_{y,2} = 1 - K_x(\eta_0 - \eta_1) + \tfrac{1}{2}\eta_0'^2,$$

$$C_{y,2} = -\frac{1}{2} g_2 \eta_0^2 - g_1 \eta_1.$$

(4) *Third order.*

$$A_{x,3} = K_x \eta_2' - K_x^2(\eta_1 \eta_0' + \eta_1' \eta_0) + K_x^3 \eta_0^2 \eta_0',$$

$$B_{x,3} = -1 + K_x(\eta_0 - \eta_1 + \eta_2) - \tfrac{3}{2} K_x \eta_0 \eta_0'^2 + 3\eta_1' \eta_0' - \tfrac{3}{2}\eta_0'^2,$$

$$C_{x,3} = \tfrac{1}{6} g_3 \eta_0^3 + g_2 \eta_1 \eta_0 + g_1 \eta_2,$$

$$B_{y,3} = -1 + K_x(\eta_0 - \eta_1 + \eta_2) - \tfrac{1}{2} K_x \eta_0 \eta_0'^2 + \eta_1' \eta_0' - \tfrac{1}{2}\eta_0'^2,$$

$$C_{y,3} = -\tfrac{1}{6} g_3 \eta_0^3 - g_2 \eta_1 \eta_0 - g_1 \eta_2.$$

APPENDIX B: HIGHER-ORDER FORMULA FOR SIMPLE CHROMATIC CONTRIBUTION

In this appendix we give the higher-order forms of the integrand G_z after performing partial integration,

$$\int_{s_0}^{s_0+L} ds_1 \beta_z(s_1) G_z(s_1) = \int_{s_0}^{s_0+L} ds_1 \hat{G}_z(s_1). \quad \text{(B1)}$$

(1) *First order.*

$$\hat{G}_{x,1} = -\beta_x(K_x^2 + g_0 - g_1 \eta_0) - 2\alpha_x K_x \eta_0' + \gamma_x K_x \eta_0, \quad \text{(B2)}$$

$$\hat{G}_{y,1} = \beta_y(g_0 - g_1 \eta_0) + \gamma_y K_x \eta_0. \quad \text{(B3)}$$

(2) *Second order.*

$$\hat{G}_{x,2} = \beta_x\Big[K_x^2\big(1 - \tfrac{1}{4}\eta_0'^2\big) - K_x^3 \eta_0 + \tfrac{1}{2} K_x^4 \eta_0^2 + g_0 - g_1(\eta_0 - \eta_1) + \tfrac{1}{2} g_2 \eta_0^2\Big] - 2\alpha_x(K_x \eta_1' - K_x^2 \eta_0 \eta_0') + \gamma_x\big(K_x \eta_1 - \tfrac{1}{2} K_x^2 \eta_0^2 + \tfrac{3}{2}\eta_0'^2\big) \quad \text{(B4)}$$

$$\hat{G}_{y,2} = -\beta_y\Big[\tfrac{1}{4} K_x^2 \eta_0'^2 + g_0 - g_1(\eta_0 - \eta_1) + \tfrac{1}{2} g_2 \eta_0^2\Big] + \gamma_y\big(K_x \eta_1 - \tfrac{1}{2} K_x^2 \eta_0^2 + \tfrac{1}{2}\eta_0'^2\big). \quad \text{(B5)}$$

(3) *Third order*.

$$\hat{G}_{x,3} = -\beta_x\Big[K_x^2\big(1 + \tfrac{1}{2}\eta_0' \eta_1'\big) - K_x^3\big\{\eta_0\big(1 + \tfrac{1}{2}\eta_0'^2\big) - \eta_1\big\} - K_x^4 \eta_0 \eta_1 + \tfrac{1}{3} K_x^5 \eta_0^3 + g_0\big(1 + \tfrac{3}{2}\eta_0'^2\big) - g_1\big\{\eta_0\big(1 + \tfrac{3}{2}\eta_0'^2\big)\eta_1 + \eta_2\big\} + g_2\big(\tfrac{1}{2}\eta_0^2 - \eta_0 \eta_1\big) - \tfrac{1}{6} g_3 \eta_0^3\Big) - 2\alpha_x[K_x \eta_2' - K_x^2(\eta_0' \eta_1 + \eta_0 \eta_1') + K_x^3 \eta_0^2 \eta_0'] + \gamma_x\Big[K_x(\eta_2 - 3\eta_0 \eta_0'^2) - K_x^2 \eta_0 \eta_1 + \tfrac{1}{3} K_x^3 \eta_0^3 + 3\eta_0' \eta_1'\Big], \quad \text{(B6)}$$

$$\hat{G}_{y,3} = \beta_y\Big[-\tfrac{1}{2} K_x^2(\eta_0'^2 + \eta_0' \eta_1') + K_x^3 \eta_0 \eta_0'^2 + g_0\big(1 + \tfrac{1}{2}\eta_0'^2\big) - g_1\big\{\eta_0\big(1 + \tfrac{1}{2}\eta_0'^2\big) - \eta_1 + \eta_2\big\} + g_2\big(\tfrac{1}{2}\eta_0^2 - \eta_0 \eta_1\big) - \tfrac{1}{6} g_3 \eta_0^3\Big] + \gamma_y\Big[K_x(\eta_2 - \eta_0 \eta_0'^2) - K_x^2 \eta_0 \eta_1 + \tfrac{1}{3} K_x^3 \eta_0^3 + \eta_0' \eta_1'\Big]. \quad \text{(B7)}$$

APPENDIX C: HIGHER-ORDER FORMULA FOR FOURIER COMPONENTS OF *G*

In this appendix we give the higher-order forms of the Fourier components of $G_{z,m}$ after performing partial integration,

$$a_{z,m}(n) = \frac{2}{\mu_{z,0}} \int_{s_0}^{s_0+L} ds_1 \cos\left[\frac{2\pi n}{\mu_{x,0}} \varphi_z(s_1)\right] \beta_z(s_1) G_{z,m}(s_1), \quad \text{(C1)}$$

$$b_{z,m}(n) = \frac{2}{\mu_{z,0}} \int_{s_0}^{s_0+L} ds_1 \sin\left[\frac{2\pi n}{\mu_{z,0}} \varphi_z(s_1)\right] \beta_z(s_1) G_{z,m}(s_1). \quad \text{(C2)}$$

(1) *First order*. After performing the partial integration, we have the explicit forms of the above integrals,

$$\begin{matrix} a_{x,1}(n) \\ b_{x,1}(n) \end{matrix} = \frac{2}{\mu_{x,0}}\int_{s_0}^{s_0+L} ds_1\Bigg[\Bigg\{\hat{G}_{x,1} - \frac{1}{2\beta_x}\left(\frac{2\pi n}{\mu_{x,0}}\right)^2 K_x\eta_0\Bigg\}\begin{matrix}\cos\\ \sin\end{matrix}\left\{\frac{2\pi n}{\mu_{x,0}}\varphi_x(s_1)\right\} \pm \frac{2\pi n}{\mu_{x,0}}\left(\frac{\alpha_x}{\beta_x}K_x\eta_0 - K_x\eta_0'\right)\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{2\pi n}{\mu_{x,0}}\varphi_x(s_1)\right\}\Bigg] \tag{C3}$$

and

$$\begin{matrix} a_{y,1}(n) \\ b_{y,1}(n) \end{matrix} = \frac{2}{\mu_{y,0}}\int_{s_0}^{s_0+L} ds_1\Bigg[\Bigg\{\hat{G}_{y,1} - \frac{1}{2\beta_y}\left(\frac{2\pi n}{\mu_{y,0}}\right)^2 K_x\eta_0\Bigg\}\begin{matrix}\cos\\ \sin\end{matrix}\left\{\frac{2\pi n}{\mu_{y,0}}\varphi_y(s_1)\right\} \pm \frac{2\pi n}{\mu_{y,0}}\frac{\alpha_y}{\beta_y}K_x\eta_0\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{2\pi n}{\mu_{y,0}}\varphi_y(s_1)\right\}\Bigg]. \tag{C4}$$

(2) *Second order*.

$$\begin{matrix} a_{x,2}(n) \\ b_{x,2}(n) \end{matrix} = \frac{2}{\mu_{x,0}}\int_{s_0}^{s_0+L} ds_1\Bigg[\Bigg\{\hat{G}_{x,2} - \frac{1}{2\beta_x}\left(\frac{2\pi n}{\mu_{x,0}}\right)^2 \times\left(K_x\eta_1 - \frac{1}{2}K_x^2\eta_0^2 + \frac{3}{2}\eta_0'^2\right)\Bigg\} \times\begin{matrix}\cos\\ \sin\end{matrix}\left\{\frac{2\pi n}{\mu_{x,0}}\varphi_x(s_1)\right\} \pm \frac{2\pi n}{\mu_{x,0}}\Bigg\{\frac{\alpha_x}{\beta_x}\left(K_x\eta_1 - \frac{1}{2}K_x^2\eta_0^2 + \frac{3}{2}\eta_0'^2\right) - K_x\eta_1' + K_x^2\eta_0\eta_0'\Bigg\}\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{2\pi n}{\mu_{x,0}}\varphi_x(s_1)\right\}\Bigg] \tag{C5}$$

and

$$\begin{matrix} a_{y,2}(n) \\ b_{y,2}(n) \end{matrix} = \frac{2}{\mu_{y,0}}\int_{s_0}^{s_0+L} ds_1\Bigg[\Bigg\{\hat{G}_{y,2} - \frac{1}{2\beta_y}\left(\frac{2\pi n}{\mu_{y,0}}\right)^2 \times\left(K_x\eta_1 - \frac{1}{2}K_x^2\eta_0^2 + \frac{1}{2}\eta_0'^2\right)\Bigg\} \times\begin{matrix}\cos\\ \sin\end{matrix}\left\{\frac{2\pi n}{\mu_{y,0}}\varphi_y(s_1)\right\} \pm \frac{2\pi n}{\mu_{y,0}}\frac{\alpha_y}{\beta_y}\left(K_x\eta_1 - \frac{1}{2}K_x^2\eta_0^2 + \frac{1}{2}\eta_0'^2\right)\begin{matrix}\sin\\ \cos\end{matrix}\left\{\frac{2\pi n}{\mu_{y,0}}\varphi_y(s_1)\right\}\Bigg]. \tag{C6}$$

APPENDIX D: CALCULATION OF FOURIER TRANSFORM

Here we use the complex Fourier transformation,

$$\beta^2(\varphi)G_1(\varphi) = \sum_{n=-\infty}^{\infty} f(n)e^{2\pi i n\varphi/\mu_0}, \tag{D1}$$

where $f(n)=[a(n)-ib(n)]/2$.

At first, we review the calculation of the double integral appearing in the second-order chromaticity,

$$\begin{aligned}
&\int_0^{\mu_0} d\varphi_2\int_0^{\varphi_2} d\varphi_1\cos(\mu_0-2\varphi_2+2\varphi_1)\beta^2(\varphi_2)\times G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)\\
&\equiv \mathrm{Re}\left[\int_0^{\mu_0} d\varphi_2\int_0^{\varphi_2} d\varphi_1 e^{i(\mu_0-2\varphi_2+2\varphi_1)}\beta^2(\varphi_2)\times G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)\right]\\
&=\mathrm{Re}\left[\sum_{n,m=-\infty}^{\infty}\frac{\mu_0^2}{2i(\mu_0+\pi m)}\left(e^{i\mu_0}\delta_{n+m,0} - \frac{\sin\,\mu_0}{\mu_0-\pi n}\right)f(n)f(m)\right].
\end{aligned}$$

The second term of the above equation is pure imaginary,

$$\begin{aligned}
&\frac{1}{2i}\mu_0^2\sin\,\mu_0\sum_{n,m=-\infty}^{\infty}\frac{f(n)}{\mu_0-\pi n}\frac{f(m)}{\mu_0+\pi m}\\
&=\frac{1}{2i}\mu_0^2\sin\,\mu_0\left|\sum_{n=-\infty}^{\infty}\frac{f(n)}{(\mu_0-\pi n)}\right|^2,
\end{aligned}$$

so that it has no contribution to the integral. Here we have used the defining identity $f(-n)=f^*(n)$ with the symbol $*$ denoting the complex conjugate. Using the same identity, we can rewrite the first term as

$$\begin{aligned}
&\int_0^{\mu_0} d\varphi_2\int_0^{\varphi_2} d\varphi_1\cos(\mu_0-2\varphi_2+2\varphi_1)\times\beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)\\
&=\sin\,\mu_0\sum_{n=-\infty}^{\infty}\frac{\mu_0^2}{2(\mu_0-\pi n)}|f(n)|^2\\
&=\sin\,\mu_0\left[\frac{1}{8}\mu_0 a_1^2(0) + \sum_{n=1}^{\infty}\frac{\mu_0^3}{4(\mu_0^2-\pi^2 n^2)}\{a_1^2(n)+b_1^2(n)\}\right].
\end{aligned}$$

Next we calculate the triple integrals in the third-order chromaticity, each of which, after the integration, becomes, respectively,

$$\begin{aligned}
&\int_0^{\mu_0} d\varphi_3\int_0^{\varphi_3} d\varphi_2\int_0^{\varphi_2} d\varphi_1\sin(\mu_0-2\varphi_3+2\varphi_1)\times\beta^2(\varphi_3)G_1(\varphi_3)\beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)\\
&=\mathrm{Im}\Bigg[\sum_{n,m,\ell=-\infty}^{\infty} e^{i\mu_0}\frac{\mu_0}{2i(\pi\ell+\mu_0)}\Bigg\{\frac{\mu_0^2}{2i[\mu_0+\pi(m+\ell)]}\times\delta_{n+m+\ell,0} - \frac{\mu_0^2}{2i(\pi n-\mu_0)}e^{-2i\mu_0}\delta_{m,0}\\
&\quad-\left(\frac{\mu_0}{2i[\mu_0+\pi(m+\ell)]}+\frac{\mu_0}{2i[\mu_0-\pi(n+m)]}\right)\times\frac{\mu_0}{2i(\pi n-\mu_0)}(e^{-2i\mu_0}-1)\Bigg\}f(n)f(m)f(\ell)\Bigg),
\end{aligned}$$

$$\int_0^{\mu_0} d\varphi_3 \int_0^{\varphi_3} d\varphi_2 \int_0^{\varphi_2} d\varphi_1 \sin(\mu_0 - 2\varphi_3 + 2\varphi_2) \times \beta^2(\varphi_3)G_1(\varphi_3)\beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)$$
$$= \mathrm{Im}\Bigg(\sum_{n,m,\ell=-\infty}^{\infty} e^{i\mu_0} \Bigg\{ - \frac{\mu_0}{2i[\pi(m+\ell)+\mu_0]} \frac{\mu_0}{2i(\pi m+\mu_0)} \times \left[\mu_0 \delta_{n+m+\ell,0} - \frac{\mu_0}{2i(\pi n - \mu_0)}(e^{-2i\mu_0}-1)\right] + \frac{\mu_0}{2\pi i \ell} \frac{\mu_0}{2i(\pi m + \mu_0)} (\mu_0 \delta_{n+m+\ell,0} - \mu_0 \delta_{n+m,0}) + \frac{\mu_0}{2i(\pi m+\mu_0)} \left[\frac{\mu_0}{2\pi i(n+m)} \mu_0 + \frac{\mu_0^2}{2} \delta_{n+m,0} \right] \delta_{\ell,0} \Bigg\} \times f(n) f(m) f(\ell) \Bigg),$$

$$\int_0^{\mu_0} d\varphi_3 \int_0^{\varphi_3} d\varphi_2 \int_0^{\varphi_2} d\varphi_1 \sin(\mu_0 - 2\varphi_2 + 2\varphi_1) \times \beta^2(\varphi_3)G_1(\varphi_3)\beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)$$
$$= \mathrm{Im}\Bigg(\sum_{n,m,\ell=-\infty}^{\infty} e^{i\mu_0} \frac{\mu_0}{2i(\pi\ell+\mu_0)} \Bigg\{ \frac{\mu_0}{2i\pi(m+\ell)} \times (\mu_0 \delta_{n+m+\ell,0} - \mu_0 \delta_{n,0}) + \left(\frac{\mu_0^2}{2\pi i n} + \frac{\mu_0^2}{2} \delta_{n,0} \right) \delta_{m+\ell,0} - \frac{\mu_0}{2i(\pi m - \mu_0)} \left[\frac{\mu_0}{2i\pi\{(n+m)-\mu_0\}} \times (e^{-2i\mu_0}-1) - \mu_0 \delta_{n,0} \right] \Bigg\} f(n) f(m) f(\ell) \Bigg).$$

Collecting the above integrals, we have

$$\int_0^{\mu_0} d\varphi_3 \int_0^{\varphi_3} d\varphi_2 \int_0^{\varphi_2} d\varphi_1 [\sin(\mu_0 - 2\varphi_3 + 2\varphi_1) - \sin(\mu_0 - 2\varphi_3 + 2\varphi_2) - \sin(\mu_0 - 2\varphi_2 + 2\varphi_1)] \times \beta^2(\varphi_3)G_1(\varphi_3)\beta^2(\varphi_2)G_1(\varphi_2)\beta^2(\varphi_1)G_1(\varphi_1)$$
$$= \frac{1}{2}\mu_0^2 \cos\mu_0 a_1(0) \left[\frac{1}{8} a_1^2(0) + \sum_{n=1}^{\infty} \frac{\mu_0^2}{4(\mu_0^2 - \pi^2 n^2)} \times \{a_1^2(n) + b_1^2(n)\} \right] - \frac{1}{16} \sin\mu_0 \Bigg[\mu_0 a_1^3(0) + 2\sum_{n=1}^{\infty} \frac{\mu_0^3(3\mu_0^2 - \pi^2 n^2)}{(\mu_0^2 - \pi^2 n^2)^2} a_1(0)\{a_1^2(n) + b_1^2(n)\} + 2\sum_{n,m=1}^{\infty} \frac{\mu_0^5\{3\mu_0^2 - \pi^2(n^2+nm+m^2)\}}{(\mu_0^2 - \pi^2 n^2)(\mu_0^2 - \pi^2 m^2)\{\mu_0^2 - \pi^2(n+m)^2\}} \times \{a_1(n)a_1(m)a_1(n+m) + a_1(n)b_1(m)b_1(n+m) + b_1(n)a_1(m)b_1(n+m) - b_1(n)b_1(m)a_1(n+m)\} \Bigg]$$

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